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RESEARCH PAPER

# Bayesian reliability-based design optimization using eigenvector dimension reduction (EDR) method

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**Abstract** In practical engineering design, most data sets for system uncertainties are insufficiently sampled from unknown statistical distributions, known as epistemic uncertainty. Existing methods in uncertainty-based design optimization have difficulty in handling both aleatory and epistemic uncertainties. To tackle design problems engaging both epistemic and aleatory uncertainties, reliability-based design optimization (RBDO) is integrated with Bayes theorem. It is referred to as Bayesian RBDO. However, Bayesian RBDO becomes extremely expensive when employing the first- or second-order reliability method (FORM/SORM) for reliability predictions. Thus, this paper proposes development of Bayesian RBDO methodology and its integration to a numerical solver, the eigenvector dimension reduction (EDR) method, for Bayesian reliability analysis. The EDR method takes a sensitivity-free approach for reliability analysis so that it is very efficient and accurate compared with other reliability methods such as FORM/SORM. Efficiency and accuracy of the Bayesian RBDO process are substantially improved after this integration.

**Keywords** Bayesian · Epistemic · RBDO · Eigenvector dimension reduction · Uncertainty

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## 1 Introduction

Reliability is of critical importance in product and process design (Hazelrigg 1998). Hence, various methods (Youn et al. 2005b; Chen et al. 1997; Du and Chen 2004) have been developed to systematically treat uncertainties in engineering analysis and, more recently, to carry out reliability-based design optimization (RBDO). In RBDO, a design optimization strategy has been advanced to improve computational efficiency and stability (Wu et al. 2001; Wang and Kodiyalam 2002; Youn et al. 2005a). Additionally, new methods for reliability assessment have been proposed to enhance numerical efficiency and stability (Du et al. 2004; Rahman and Xu 2004a; Youn et al. 2006b). In practical engineering applications, the amount of uncertainty data is extremely restricted mainly due to limited resources (e.g., manpower, expense, time). Generally, a random (or uncertain) variable where its random property is completely known is defined as an "aleatory" random variable, whereas a random variable with an insufficient amount of data where its random property is incompletely known is defined as an "epistemic" random variable. To deal with epistemic uncertainty, Bayesian approach has been investigated for the reliability modeling (Zhang and Mahadevan 2000), and in this paper, Bayesian reliability-based design optimization (Bayesian RBDO) methodology is proposed to deal with engineering design problems, which involve both aleatory and epistemic uncertainties. Bayesian RBDO is a general reliability-based design method that embraces traditional RBDO as a special case. The capability to deal with both aleatory and epistemic uncertainties is of vital importance in practical engineering applications constrained by limited resources.

However, Bayesian RBDO could be expensive if the first- or second-order reliability method (FORM/SORM) is used for reliability analysis. This is because the numerical algorithms in FORM/SORM require sensitivity and/or Hessian information of system responses during the reliability analysis process (Madsen et al. 1986; Palle and Michael 1982). Many algorithms have been proposed to improve efficiency and accuracy of reliability analysis (Youn et al. 2003; Wu et al. 1990; Wu 1994; Hasofer and Lind 1974; Liu and Kiureghian 1991; Wang and Grandhi 1994; Wang and Grandhi 1996). To overcome this challenge, this paper proposes an integration of Bayesian RBDO with the eigenvector dimension reduction (EDR) method (Youn et al. 2007; Zhimin et al. 2007). The EDR method additively decomposes the multidimensional integration of the probability function into a series of one-dimensional integrations to increase the computational efficiency while maintaining good accuracy. Then, the EDR method computes statistical moments of the system responses, which are used to generate the probability density functions (PDFs) of the system responses. The EDR method facilitates the Bayesian RBDO process with high efficiency and accuracy.

In this paper, we propose development of Bayesian RBDO methodology and its integration to a numerical solver, the EDR method, for Bayesian reliability analysis. Five technique contributions are made in this paper: (1) Bayesian approach is successfully applied on RBDO, (2) Bayesian reliability is uniquely defined for design optimization, (3) a guideline of target Bayesian reliability is made as a function of data size, (4) sensitivity analysis is developed for Bayesian RBDO, and (5) the EDR method is employed for Bayesian reliability analysis. Section 2 of this paper briefly introduces RBDO under both aleatory and epistemic uncertainties, and Section 3 presents the proposed Bayesian RBDO methodology. Section 4 briefly describes the EDR method for reliability analysis. One example is used to demonstrate the effectiveness of the EDR method in terms of efficiency and accuracy for reliability analysis. The integration of Bayesian RBDO with the EDR method will be discussed in Section 5. The proposed method is applied to one mathematical example and an engineering case study in Section 6.

## 2 Introduction of RBDO under both aleatory and epistemic uncertainties

Knowing that both aleatory and epistemic uncertainties exist in the system of interest, RBDO can be formulated as

$$\begin{aligned} &\text{minimize} && C(\mathrm{X_a}, \mathrm{X_e}; \mathrm{d}) \\ &\text{subject to} && P_B(G_i(\mathrm{X_a}, \mathrm{X_e}; \mathrm{d}) \leq 0) \geq \Phi(\beta_{t_i}), \quad i = 1, \cdots, np \\ & && \mathrm{d}^{\mathrm{L}} \leq \mathrm{d} \leq \mathrm{d}^{U}, \quad \mathrm{d} \in R^{nd} \text{ and } \mathrm{X_a} \in R^{na}, X_e \in R^{ne} \end{aligned} \tag{1}$$

where $P_B$ $(G_i(\mathbf{X}_a,\mathbf{X}_e;\mathbf{d})\leq 0)=R_i^B(\mathbf{X}_a,\mathbf{X}_e;\mathbf{d})$ is Bayesian reliability where $G_i(\mathbf{X}_a,\mathbf{X}_e;\mathbf{d})\leq 0$ is defined as a safety event; $C$ $(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d})$ is the objective function; $\mathbf{d}=\mu(\mathbf{X})$ is the design vector; $\mathbf{X}_a$ and $\mathbf{X}_e$ are the aleatory and epistemic random vectors, respectively; $\beta_t$ is a prescribed target reliability index; and $np$, $nd$, $na$, and $ne$ are the numbers of probabilistic constraints, design variables, aleatory random variables, and epistemic random variables, respectively. Among all (both aleatory and epistemic) random variables, if the parameters describing a random variable are controllable, they are considered as design variables. For instance, a random variable with a normal distribution may have two design variables, mean and standard deviation.

When modeling epistemic uncertainties with insufficient data, the degree of statistical uncertainty could be greater than that of physical uncertainty. Then, although RBDO is successfully performed, the optimum design could be unreliable due to a lack of data. Hence, when involving epistemic uncertainties, the probabilistic constraints in a Bayesian sense, as shown in (2), will ensure to produce a reliable design.

$$R_i^B(\mathbf{X}_\mathrm{a}, \mathbf{X}_\mathrm{e}; \mathbf{d}) = \int \underset{G_i(\mathrm{X_a X_e}) \leq 0}{\cdots} \int f_{\mathrm{x_a x_e}}(\mathbf{x}_\mathrm{a}\mathbf{x}_\mathrm{e}) d\mathbf{x}_\mathrm{a} d\mathbf{x}_\mathrm{e} \geq \Phi(\beta_{t_i}) \geq R_i^t \tag{2}$$

where $R_i^B$ $(\mathbf{X}_a,\mathbf{X}_e;\mathbf{d})$ is the Bayesian reliability, $R_i^t = \Phi(\beta_{t_i})$ is a target reliability, and $f_{\mathbf{X}_\mathrm{a}\mathbf{X}_\mathrm{e}}(\mathbf{x}_\mathrm{a}\mathbf{x}_\mathrm{e})$ is the joint PDF of both aleatory and epistemic uncertainties. Due to lacking data for epistemic uncertainty, it is impossible to precisely model the joint PDF $f_{\mathbf{X}_e}(\mathbf{x}_e)$ for epistemic uncertainties. Modeling the joint PDF for epistemic uncertainties strongly depends on a set of data and is thus subjective. For example, let $X_1$ and $X_2$ be epistemic and aleatory uncertainties, respectively. $X_2$ is assumed to be normally distributed with $N(\mu=5.0, \sigma=1.0)$, whereas $X_1$ is modeled with 20 samples generated from $N$ (5.0, 1.0). The mean, standard deviation, skewness, and kurtosis from the data are 4.5789, 0.9294, 0.2908, and 2.4851, respectively. It is found that the Gamma distribution, $f(x_2)=\mathrm{b}^{-a}\Gamma^{-1}(a)x^{a-1}e^{-x/b}$, provides the best fit to the data with $a=25.4439$ and $b=0.1800$. By assuming that both random variables are statistically independent, the joint PDF can be plotted, as shown in Fig. 1.

As shown in Fig. 1, inaccurate joint PDF could lead to inaccurate reliability estimate and unreliable design due to insufficient data. Therefore, the presence of the epistemic uncertainty in the formulation prohibits the use of conventional methods to calculate the probabilistic constraints, and this necessitates the Bayesian approach for reliability assessment and reliability-based design. RBDO turns to be a special case of Bayesian RBDO because Bayesian RBDO is able to handle aleatory and/or epistemic uncertainties. The next section presents the definition of Bayesian reliability and a method for Bayesian reliability analysis.

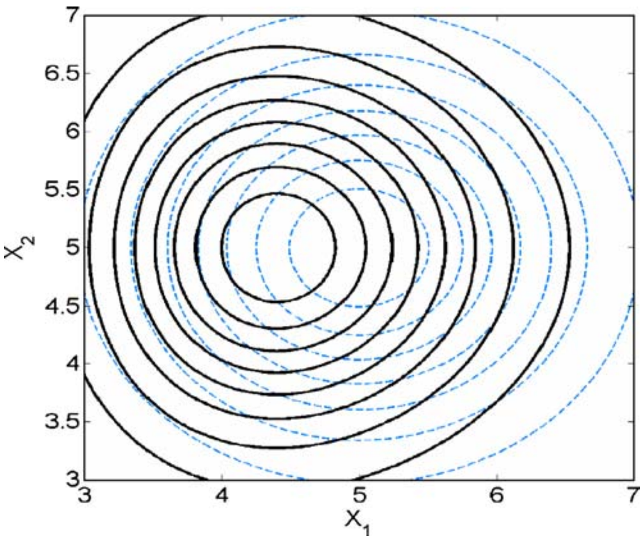


**Fig. 1** Joint PDF contours: *black* (approximate joint PDF with limited data) and *blue* (true joint PDF)

## 3 Bayesian RBDO

### 3.1 Bayesian binomial inference and reliability distribution

When modeling uncertainties with insufficient data, reliability must be uncertain and subjective. A question is how to model uncertain and subjective reliability in a probability sense. The following discussion will answer the question of modeling reliability using the Bayesian binomial inference.

In many engineering applications, outcomes of events from repeated trials can be a binary manner, such as occurrence or nonoccurrence, success or failure, good or bad, and so forth. In such cases, random behavior can be modeled with a discrete probability distribution model. In addition, if the events satisfy the additional requirements of a Bernoulli sequence, that is to say, if the events are statistically independent and the probability of occurrence or nonoccurrence of events remains constant, they can be mathematically represented by the *binomial distribution* (Haldar and Mahadevan 2000). In other words, if the probability of an event occurrence in each trial is $p$ and the probability of nonoccurrence is $(1-r)$, then the probability of $x$ occurrences out of a total of $N$ trials can be described by the probability mass function (PMF) of a Binomial distribution as

$$\Pr(X = x, N|r) = \binom{N}{x} r^x (1-r)^{N-x} \quad x = 0, 1, 2, \ldots, N \tag{3}$$

where the probability of success identified in the previous test, $r$, is the parameter of the distribution.

In (3), a given $r$ is provided before the calculation of the probability of $x/N$ ($x$ occurrences out of $N$ trials). However, when $r$ is an uncertain parameter and a prior distribution is provided, the Bayesian inference process can be employed to update $r$ based on the outcomes of the trials. Given $x$ occurrences out of a total of $N$ trials, the probability distribution of $r$ can be calculated using Bayes' rule as (Li et al. 2002)

$$f(r|x) = \frac{f(r)f(x|r)}{\int_0^1 f(r)f(x|r)dr} \tag{4}$$

where $f\,(r)$ is the prior distribution of $r$, $f\,(r|x)$ is the posterior distribution of $r$, and $f\,(x|r)$ is the likelihood of $x$ for a given $r$. The integral in the denominator is a normalizing factor to make the probability distribution proper. The prior distribution is known for $r$, prior to the current trials. In this paper, a uniform prior distribution is used to model $r$ bounded in [0, 1]. However, it is possible to obtain a posterior distribution with any type of a prior distribution.

For Bayesian reliability predictions, both prior reliability distribution ($r$) and the number ($x$) of safety occurrences out of the total number of test data set $N$ must be known. If prior reliability distribution ($r$) is unavailable, it will be simply modeled with a uniform distribution, $r \sim U\,(a, b)$ where $a<b$ and $a, b \in [0, 1]$. At an early design stage, it can be modeled using reliability for the previous product designs or expert opinions. If the reliability distribution has been predicted with a data set in a precedent test, this reliability distribution will be used as the prior reliability distribution and updated to posterior reliability distribution with a new data set. In all cases, reliability will be modeled with Beta distribution, the conjugate distribution of the Bayesian binomial inference, because the uniform distribution is a special case of the Beta distribution. Equation (5) is the PDF of the Beta distribution as

$$f(r|x) = \frac{1}{B(\alpha, \beta)} r^{\alpha-1}(1-r)^{\beta-1}, \quad \left( B(\alpha,\ \beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt \right) \tag{5}$$

where $\alpha=x+1$ and $\beta=N-x+1$. The posterior distribution, $f\,(p|x)$, is the Beta distribution and represents the probability distribution of reliability. It is found that the distribution is a function of $x$ and $N$, the number of safety trials and the total number of trials, respectively.

Figures 2 and 3 show such a functional relationship between the Bayesian reliability distribution and its parameters, $x$ and $N$. Figure 2 demonstrates the dependence of the reliability PDF on the number of safety occurrences, $x$, out of the given $N$ trials (e.g., $N$=40 in Fig. 2). The larger the number of safety occurrences for a given $N$ trials, the greater the mean of reliability. The PDF of reliability appears to be feasible because the mean of the PDF is close to $x/N$, which is a Frequentist estimate of reliability (e.g., $\mu_{\text{Beta}(5,37)} \approx 4/36$). Figure 3 exhibits the dependence of

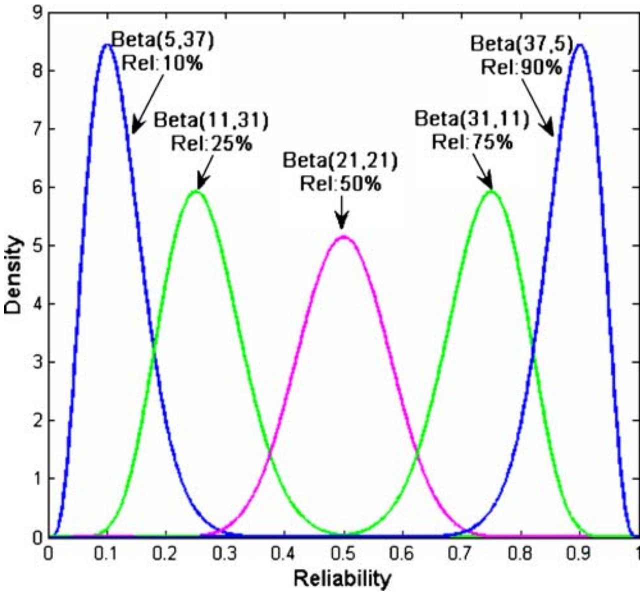


**Fig. 2** Dependence of the PDF of reliability on the number of safety occurrences, $x$

the reliability PDF on the total number of trials ($N$) with the same ratio of $x$ to $N$. As the total number of trials is increased, the variation of reliability is decreased, such as $\sigma_{\mathrm{Beta}(451,151)} < \sigma_{\mathrm{Beta}(151,51)} < \sigma_{\mathrm{Beta}(46,16)} < \sigma_{\mathrm{Beta}(16,6)}$. In other words, the PDF of reliability asymptotically converges to the exact reliability with the increase of the number of trials. It is also associated with a confidence of the reliability model. The larger the data size, the higher the confidence level of the reliability model. This will also be observed in Bayesian RBDO in Section 5.1. In summary, this posterior distribution is feasible and applicable to model reliability, despite the dearth of data for modeling uncertainties. The Bayesian updating process can be conveniently carried out because the Beta distribution for uncertain reliability, $r$, is the conjugate distribution.

## 3.2 PDF of reliability with both aleatory and epistemic uncertainties

When only epistemic uncertainties are engaged to assess reliability, its PDF can be modeled using the Beta distribution in (5) by counting the number of safety occurrences, $x$. In general, both aleatory and epistemic uncertainties generally appear in most engineering design problems. In such situations, the PDF of reliability can be similarly obtained through Bayesian reliability analysis. To build the PDF of reliability, reliability analysis must be performed at every data point for epistemic uncertainties while considering aleatory uncertainties. Different reliability measures, $R_k=R(x_{e,k})$, are obtained at different sample points for epistemic uncertainties. In (5), $\alpha=x+1$ and $\beta=N-x+1$, where $x=\Sigma R_k$. Then, the PDF of reliability $r$ with a uniform prior distribution is updated to $R(\mathbf{X}_a, \mathbf{X}_b; \mathbf{d})$ as

$$R(\mathbf{X}_{\mathrm{a}}, \mathbf{X}_{\mathrm{e}}; \mathbf{d}) = f(r|\overline{\mathbf{x}}) = \frac{1}{Beta(\alpha,\beta)} r^{\mathrm{a}-1}(1-r)^{\beta-1}$$
$$\text{where } \alpha = 1+x, \quad \beta = N-x+1, \quad \overline{\mathbf{x}} = \left\{\mathbf{x}_{e,1}, \cdots, \mathbf{x}_{e,N}\right\}$$
$$x = \sum R_k, \text{ and } R_k = \Pr\left[g(X_{\mathrm{a}}) \le 0 | \mathbf{x}_{e,k}\right] \tag{6}$$

$N$ is the number of finite data sets for epistemic uncertainties. A detailed description is given in the following example.

## 3.3 Definition of Bayesian reliability

For design optimization, Bayesian reliability must satisfy two requirements: (a) sufficiency and (b) uniqueness. The sufficiency requirement means that the Bayesian reliability must be smaller than an exact reliability realized with a sufficient amount of data for the input uncertainties. Then, Bayesian RBDO provides an optimum design with higher reliability than target reliability, regardless of the data size. To meet the sufficiency requirement, an extreme distribution theory for the smallest reliability value is employed to guarantee the sufficiency of reliability. $R$ values are different for different data sets, $\mathbf{x}_{e,k}$, of which each has the same sample size $N$. Without generating expensive data sets, the extreme distribution theory determines the probability distribution of the smallest $R$ value that guarantees the first requirement. Then, the median value of the extreme distribution uniquely determines Bayesian reliability. To satisfy both requirements, Bayesian reliability is defined as the median value of the extreme distribution for the smallest value derived from the Beta distribution in (6).

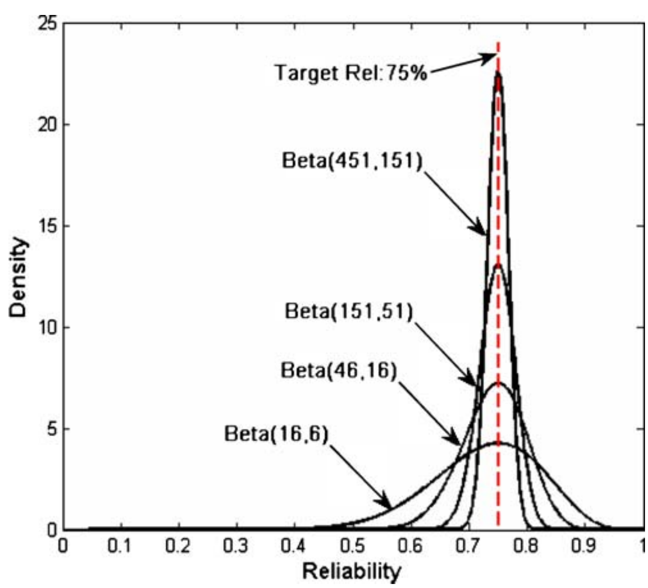


**Fig. 3** Dependence of the PDF of reliability on the total number of trials, $N$

**Table 1** $X_1$ samples and probabilities

| $X_1$ | Probability |
|---|---|
| 3.1047 | 0.99986 |
| 2.7750 | 0.84471 |
| 2.8175 | 0.91969 |
| 2.9277 | 0.99018 |
| 3.1706 | 0.99997 |
| 3.4741 | 1.00000 |
| 2.9029 | 0.98353 |
| 2.8196 | 0.92247 |
| 2.7157 | 0.67025 |
| 2.8869 | 0.97730 |
| 2.7605 | 0.80983 |
| 3.1006 | 0.99984 |
| 2.5933 | 0.19228 |
| 2.9604 | 0.99520 |
| 2.9354 | 0.99168 |
| 2.9575 | 0.99488 |
| 2.9430 | 0.99295 |
| 2.9706 | 0.99618 |
| 2.6738 | 0.50480 |
| 2.8185 | 0.92101 |

First, based on the extreme distribution theory, the extreme distribution for the smallest reliability value is constructed from the reliability distribution, Beta distribution. For random reliability $R$ with the Beta distribution function, $F_R(r)$, let ${}^1R$ be the smallest value among $N$ data points for random reliability, $R$. Then, the cumulative distribution function (CDF) of the smallest reliability value, ${}^1R$, can be expressed as (Rao 1992)

$$\begin{aligned} 1 - F_{{}^1R}(r) &= P({}^1R > r) \\ &= P({}^1R > r, {}^2R > r, \cdots, {}^NR > r) \end{aligned} \tag{7}$$

Because the $i$th smallest reliability values, ${}^iR$ ($i$=1,..., $N$), are identically distributed and statistically independent, the CDF of the smallest reliability value becomes

$$F_{{}^1R}(r) = 1 - [1 - F_R(r)]^N \tag{8}$$

Bayesian reliability, $R_B$, is defined as the median value of the reliability distribution. That is to say, Bayesian reliability is the solution of the nonlinear equation in (8) by setting $F_{{}^1R}(R_B) = 0.5$.

$$R_B = F_R^{-1}\left[1 - \sqrt[N]{1 - F_{{}^1R}(r^m)}\right] = F_R^{-1}\left[1 - \sqrt[N]{0.5}\right] \tag{9}$$

3.4 Numerical procedure of Bayesian reliability analysis

Bayesian reliability analysis can be conducted using the following numerical procedure as

Step 1 Collect a limited data set for epistemic uncertainties where the data size is $N$.

Step 2 Calculate reliabilities ($R_k$) with consideration of aleatory uncertainties at all epistemic data points.

Step 3 Build a distribution of reliability using the Beta distribution in (6) with aleatory and/or epistemic uncertainties.

Step 4 Construct the extreme distribution in (8) with the Beta distribution obtained in step 3.

Step 5 Determine the Bayesian reliability using (9).

A mathematical example is used to help understand the numerical procedure of Bayesian reliability analysis.

*Example* Let $g(X_1,X_2)=1-X_1^2X_2/20\leq 0$ be an inequality constraint with two random variables where $X_1$ is an epistemic random variable and $X_2$ is an aleatory random variable, $X_2 \sim N$ ($\mu_2$=2.8, $\sigma_2$=0.2). To calculate the actual reliability so as to show the comparison of the modeled reliability distribution with the actual reliability, although $X_1$ is an epistemic variable and its distribution is assumed to be unknown, 20 data for $X_1$ are randomly sampled from *Normal* distribution ($\mu_1$=2.9, $\sigma_1$=0.2), as shown in Table 1. The table also shows the corresponding reliabilities $R_k$= Pr $[g(X_2)\leq 0|X_1(k)]$ for $k$=1,..., 20 that are computed from reliability analyses. For example, $X_1(1)$=3.1047, then $R_1$= $P$ $[1\text{–}3.1047^2 * X_2/20\leq 0]$=0.99986. From Table 1, the expected number of safe design points out of the 20 designs can be obtained from the sum of all 20 reliabilities, $x=\Sigma R_k$= 17.7066. The reliability can then be modeled by the Beta distribution as *Beta*(18.7066, 3.2934) at the design point, ($\mu_1$=2.9, $\mu_2$=2.8). This is graphically shown in Fig. 4.

To validate the results, $X_1$ is assumed to follow $N(\mu_1$= 2.9, $\sigma_1$=0.2). A Monte Carlo simulation (1,000,000 samples) obtains the actual reliability (which is 0.8488) of the design point. As shown in Fig. 4, the actual reliability is close to the mean value of the reliability distribution.

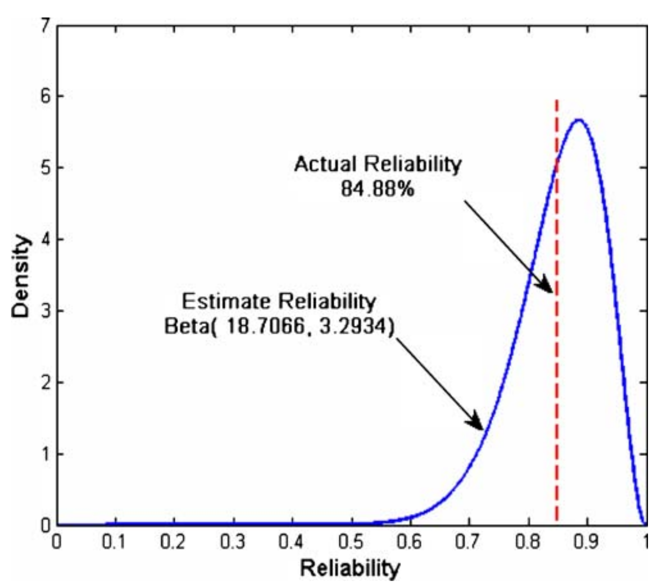


**Fig. 4** Actual and estimated reliability

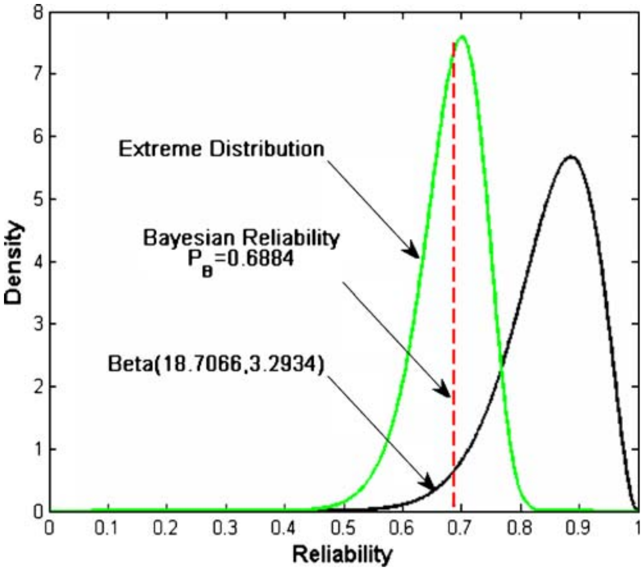


**Fig. 5** Bayesian reliability

Therefore, the reliability distribution gives a quite feasible estimate with both aleatory and epistemic uncertainties. In this example, a uniform distribution, $r\sim U(0,1)$, is used as the prior distribution of reliability. Therefore, the reliability distribution appears to be widely distributed, but it can be narrowly distributed if the prior distribution is more precisely given.

Using (8), the extreme distribution for the smallest reliability value is obtained as

$$F_{{}^{1}R}(r) = 1 - \left[1 - \int_0^{{}^{1}R} \frac{1}{B(18.7066, 3.2934)} \theta^{17.7066}(1-\theta)^{2.2934}\right]^{20}$$

From (9), Bayesian reliability is calculated as $P_B$= 0.6884. The Beta distribution for reliability, its extreme distribution for the smallest reliability value, and the Bayesian reliability are graphically shown in Fig. 5.

### 3.5 A setup of Bayesian target reliability

Target reliability must depend on the data size of epistemic uncertainties. With few data for uncertainties, setting target reliability to 99.9% is not possible. Although high reliability is achieved through RBDO, the confidence of reliability will be extremely low. This section provides a guideline to set target reliability in Bayesian RBDO, which depends on a data size of epistemic uncertainties. To assist the setup of target reliability, the maximum Bayesian reliability, $P_B^{\max}$, will be determined for a given sample size, $N$. For example, suppose that the prior distribution is a uniform distribution, say, $r\sim U\,(0,1)$. The Beta distribution for reliability is $Beta\,(1+N,1)$ with all safe samples. Then, the maximum Bayesian reliability can be defined for a given sample size as

$$P_B^{\max} = \mathrm{median}[F_{{}^{1}R}(r)] \quad \text{where } R\sim Beta(1+N,1) \tag{10}$$

This is graphically shown in Fig. 6. As the data size rises, $P_B^{\max}$ rapidly increases to 90% and then slowly increases. Target reliability must be set lower than the maximum Bayesian reliability for a given data size. For example, the target reliability with 50 data for epistemic uncertainties must be lower than 92%. If the prior distribution is more precisely given, such as $Beta(\alpha, \beta)$, the distribution for reliability in (10) can be generally set to $R\sim Beta(\alpha+N, \beta)$. Then, $P_B^{\max}$ can be correspondingly obtained.

### 3.6 Formulation of Bayesian RBDO

Knowing that both aleatory and epistemic uncertainties exist in the system of interest, Bayesian RBDO can be formulated as

$$\begin{array}{ll} \text{minimize} & C(\mathbf{X}_{\mathrm{a}}, \mathbf{X}_e; \mathbf{d}) \\ \text{subject to} & P_B(G_i(\mathbf{X}_{\mathrm{a}}, \mathbf{X}_e; \mathbf{d}) \le 0) \ge \Phi(\beta_{t_i}), \quad i = 1, \cdots, np \\ & \mathbf{d}^{\mathrm{L}} \le \mathbf{d} \le \mathbf{d}^{\mathrm{U}}, \quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X}_{\mathrm{a}} \in R^{na}, \mathbf{X}_e \in R^{ne} \end{array} \tag{11}$$

where $P_B(G_i(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d})\le 0)=R_{B,i}$ is Bayesian reliability where $G_i(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d})\le 0$ is defined as a safety event; $C(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d})$ is the objective function; $\mathbf{d}=\boldsymbol{\mu}(\mathbf{X})$ is the design vector; $\mathbf{X}_a$ and $\mathbf{X}_e$ are the aleatory and epistemic random vectors, respectively; $\beta_{ti}$ is a prescribed target Bayesian reliability index; and $np$, $nd$, $na$, and $ne$ are the numbers of probabilistic constraints, design variables, aleatory random variables, and epistemic random variables, respectively.

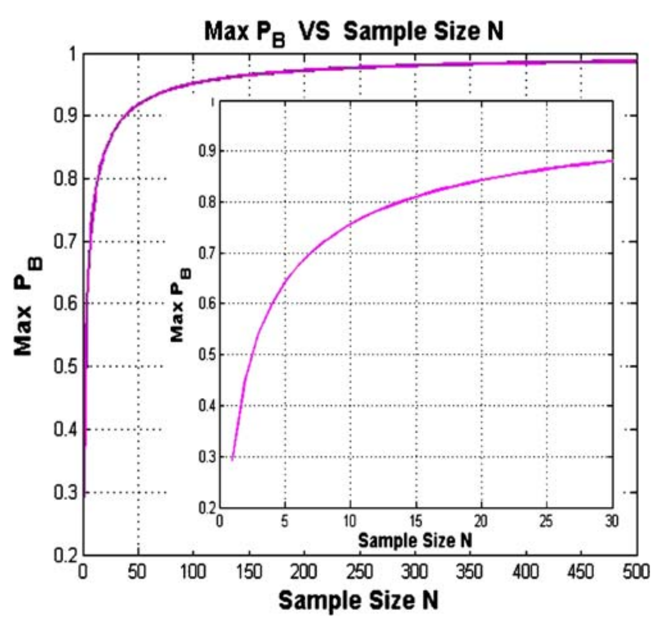


**Fig. 6** Target Bayesian reliability with sample size, $N$

### 3.7 Sensitivity analysis of Bayesian reliability

Let the $i$th reliability follow the Beta distribution, Beta $(\alpha_i,\beta_i)$, where

$$\begin{aligned}\alpha_i &= 1 + x = 1 + R_1 + R_2 + \cdots + R_N \\ \beta_i &= 2 + N - \alpha_i\end{aligned} \tag{12}$$

Recall the definition of Bayesian reliability in (9).

$$F_{R_i}\left(R_i^B\right) = 1 - \sqrt[N]{0.5} = \int_0^{R_i^B} \frac{1}{B(\alpha_i,\beta_i)}\theta^{\alpha_i-1}(1-\theta)^{\beta_i-1}d\theta \tag{13}$$

From (13), a function $g(R_i^B, \alpha_i)$ can be defined as

$$g\left(R_i^B,\alpha_i\right) = \int_0^{R_i^B} \frac{1}{B(\alpha_i, 2+N-\alpha_i)}\theta^{\alpha_i-1}(1-\theta)^{1+N-\alpha_i}d\theta + \sqrt[N]{0.5} - 1 \tag{14}$$

The sensitivity for Bayesian reliability with respect to $d_j$ can be expressed as

$$\frac{\partial R_i^B}{\partial d_j} = \frac{dR_i^B}{d\alpha_i}\cdot\frac{\partial\alpha_i}{\partial d_j} \tag{15}$$

where

$$\frac{dR_i^B}{d\alpha_i} = -\frac{\partial g_i/\partial\alpha_i}{\partial g_i/\partial R_i^B} \tag{16}$$

$$\frac{\partial\alpha_i}{\partial d_j} = \frac{\partial P_1}{\partial d_j} + \frac{\partial P_2}{\partial d_j} + \cdots + \frac{\partial P_N}{\partial d_j} \tag{17}$$

After calculating two parts in the right side of (15) through (16) and (17), the sensitivity for Bayesian reliability with respect to the $j$th design variable, $d_j$, can be expressed as

$$\frac{\partial R_i^B}{\partial d_j} = -\frac{\int_0^{R_i^B}\theta^{\alpha_i-1}(1-\theta)^{1+N-\alpha_i}\left(\ln\frac{\theta}{1+\theta} + B_i\frac{dBe_i^{-1}}{d\alpha_i}\right)d\theta}{\left(R_i^B\right)^{\alpha_i-1}\left(1-R_i^B\right)^{1+N-\alpha_i}}\sum_{j=1}^{N}\frac{\partial P_j}{\partial d_j} \tag{18}$$

Note that the integration part in (18) may encounter numerical singularity when computing the sensitivity of Bayesian reliability. Additionally, it follows a complicated mathematical derivation and implementation. A more simple way is sought to calculate Bayesian reliability sensitivity. The idea comes from a one-to-one mapping between Bayesian reliability and the mean value of the Beta distribution (the posterior distribution) for reliability for a given sample size, $R_i^B = R_i^B(Mi)$ or $M_i = M_i\left(R_i^B\right)$. The transform between these two values is shown in (19):

$$1 - \sqrt[N]{0.5} = \int_0^{R_i^B}\frac{1}{B(p,q)}\theta^p(1-\theta)^q d\theta \tag{19}$$

where $p=(N+2)M_{\mathrm{i}}$, $q=(N+2)(1-M_{\mathrm{i}})$. Instead of Bayesian reliability, the corresponding mean value of the beta distribution for reliability and the sensitivity of the mean value with respect to the design variables will be used for design optimization. For a given sample size, the one-to-one mapping relates a target Bayesian reliability to a single-valued target mean value of the beta distribution for reliability. Thus, satisfaction of the target mean value of the beta distribution for reliability always ensures satisfaction of the target Bayesian reliability.

Suppose that the Beta distribution Beta $(\alpha, \beta)$ is used to model reliability. Its mean value, $M_i = M_i\left(R_B^i\right)$, can be expressed as

$$M_i = \frac{\alpha_i}{\alpha_i+\beta_i} = \frac{\alpha_i}{N+2} \tag{20}$$

The sensitivity of its mean value to design variable, $d_j$, can be expressed as

$$\frac{\partial M_i}{\partial d_j} = \frac{1}{N+2}\frac{\partial\alpha_i}{\partial d_j} \tag{21}$$

From (17), (21) can be expressed as

$$\frac{\partial M_i}{\partial d_j} = \frac{1}{N+2}\left(\frac{\partial P_1}{\partial d_j} + \frac{\partial P_2}{\partial d_j} + \cdots + \frac{\partial P_N}{\partial d_j}\right) \tag{22}$$

Results of reliability mean, $M_i = M_i\left(R_B^i\right)$, can be converted to a reliability index. Then, the sensitivity can be developed for the format of the reliability index $\beta_i^B$, where $\beta_i^B = \Phi^{-1}(M_i)$. Correspondingly, all reliabilities, $P_i, i = 1{\sim}N$, can be transformed into the reliability indices, $\beta_i$. *The sensitivity of Bayesian reliability index can be expressed as*

$$\frac{\partial\beta_i^B}{\partial d_j} = \frac{\partial M_i}{\partial d_j}\Bigg/\frac{\partial M_i}{\partial\beta_i^B} \tag{23}$$

where

$$\frac{\partial M_i}{\partial\beta_i^B} = \frac{1}{\sqrt{2\pi}}e^{-\frac{\left(\beta_i^B\right)^2}{2}} \tag{24}$$

Similarly,

$$\frac{\partial M_i}{\partial d_j} = \frac{1}{N+2}\left(\frac{\partial P_1}{\partial\beta_1}\frac{\partial\beta_1}{\partial d_j} + \cdots + \frac{\partial P_N}{\partial\beta_N}\frac{\partial\beta_N}{\partial d_j}\right) \tag{25}$$

where

$$\frac{\partial P_i}{\partial d_j} = \frac{\partial P_i}{\partial\beta_i}\frac{\partial\beta_i}{\partial d_j} = \frac{1}{\sqrt{2\pi}}e^{-\frac{\beta_i^2}{2}}\frac{\partial\beta_i}{\partial d_j} \tag{26}$$

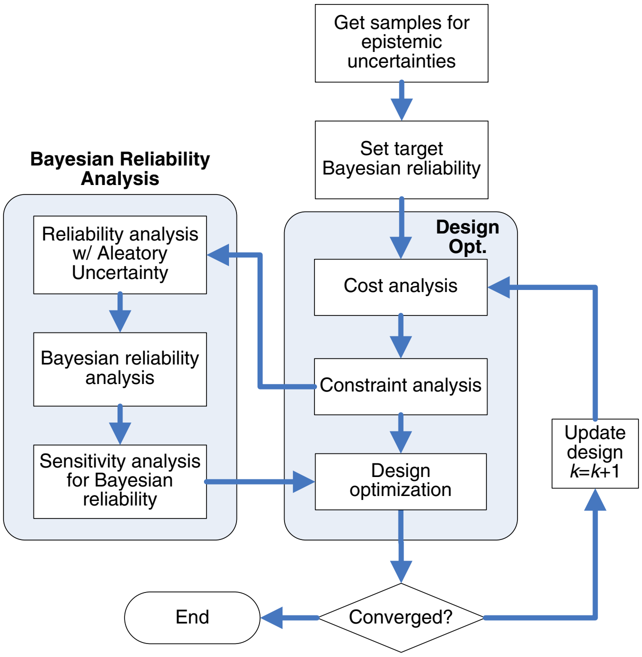


**Fig. 7** Bayesian RBDO flowchart

By substituting the sensitivity of reliability index, $\partial\beta_i/\partial d_j$, into (26), the sensitivity of Bayesian reliability, $\partial\beta_i^B/\partial d_j$, can be obtained as

$$\frac{\partial \beta_i^B}{\partial d_j} = \frac{e^{\frac{\left(\beta_i^B\right)^2}{2}}}{N+2}\sum_{k=1}^{N} e^{-\frac{\beta_k^2}{2}}\frac{\partial \beta_k}{\partial d_j} \tag{27}$$

where $i$=1,...,$nc$, $j$=1,...,$nd$, $k$=1,...,$N$.

### 3.8 Numerical procedure of Bayesian RBDO

Based on the discussion in the previous sections, the Bayesian RBDO procedure is presented in Fig. 7. The Bayesian reliability analysis in the left shaded box in Fig. 6 calculates the Bayesian reliabilities, as well as their sensitivities, which require reliability analyses at all epistemic sample points. For instance, the probabilistic constraints at any data point for epistemic uncertainties become functions of only aleatory uncertainties, and then the existing reliability analysis methods (FORM, SORM, or EDR method, etc.) are used for reliability and its sensitivity analyses. Thus, one Bayesian reliability analysis engages reliability and its sensitivity analyses $N$ times. This is why Bayesian RBDO could become expensive, and thus, more investigation must be made to reduce its computational effort. Once the cost function, Bayesian reliability, and their sensitivities are computed, design optimization is conducted in the right shaded box in Fig. 7. It is clear from the flowchart that Bayesian RBDO completely integrates Bayesian reliability analysis to RBDO.

**Table 2** Properties of random variables of vehicle side-impact model

| Random variables | Distr. type | Std. dev. |
|---|---|---|
| $X_1$ (B-pillar inner) | Normal | 0.050 |
| $X_2$ (B-pillar reinforce) | Normal | 0.050 |
| $X_3$ (floor side inner) | Normal | 0.050 |
| $X_4$ (cross member) | Normal | 0.050 |
| $X_5$ (door beam) | Normal | 0.050 |
| $X_6$ (door belt line) | Normal | 0.050 |
| $X_7$ (roof rail) | Normal | 0.050 |
| $X_8$ (mat. B-pillar inner) | Normal | 0.006 |
| $X_9$ (mat. floor side inner) | Normal | 0.006 |
| $X_{10}$ (barrier height) | Normal | 10.0 |
| $X_{11}$ (barrier hitting) | Normal | 10.0 |

**Table 3** Components and safety criteria of vehicle side-impact model

| Components | Safety criteria |
|---|---|
| $G_1$: abdomen load (kN) | ≤1 |
| $G_2$–$G_4$: rib deflection | |
| Upper/middle/lower | ≤32 |
| $G_5$–$G_7$: VC (m/s) | |
| Upper/middle/lower | ≤10.32 |
| $G_8$: pubic symphysis force (kN) | ≤4 |
| $G_9$: velocity of B-pillar | ≤99 |
| $G_{10}$: velocity of front door at B-pillar | ≤15.7 |

## 4 Eigenvector dimension reduction method for reliability analysis

In general, statistical moments (or PDF) of system responses can be calculated as

$$E\{Y^m(\mathbf{X})\} = \int_{-\infty}^{+\infty}\cdots\int_{-\infty}^{+\infty} Y^m(\mathbf{x})\cdot f_{\mathbf{X}}(\mathbf{x})\cdot d\mathbf{x} \tag{28}$$

In (28), a major challenge here is the multidimensional integration over the entire random input domain. To resolve this difficulty, the EDR method (Youn et al. 2007; Zhimin et al. 2007) uses an additive decomposition (Rahman and

**Table 4** Results of component reliability analysis

| Const. | FORM | SORM | EDR | MCS |
|---|---|---|---|---|
| $G_1$ | 1 | 1 | 1 | 1 |
| $G_2$ | 1 | 1 | 1 | 1 |
| $G_3$ | 0.9989 | 0.9989 | 0.9989 | 0.9989 |
| $G_4$ | 0.9 | 0.9136 | 0.9026 | 0.9026 |
| $G_5$ | 1 | 1 | 1 | 1 |
| $G_6$ | 1 | 1 | 1 | 1 |
| $G_7$ | 1 | 1 | 1 | 1 |
| $G_8$ | 0.9 | 0.8723 | 0.7097 | 0.7019 |
| $G_9$ | 0.9897 | 0.9905 | 0.9905 | 0.99 |
| $G_{10}$ | 0.9 | 0.9025 | 0.9495 | 0.9444 |

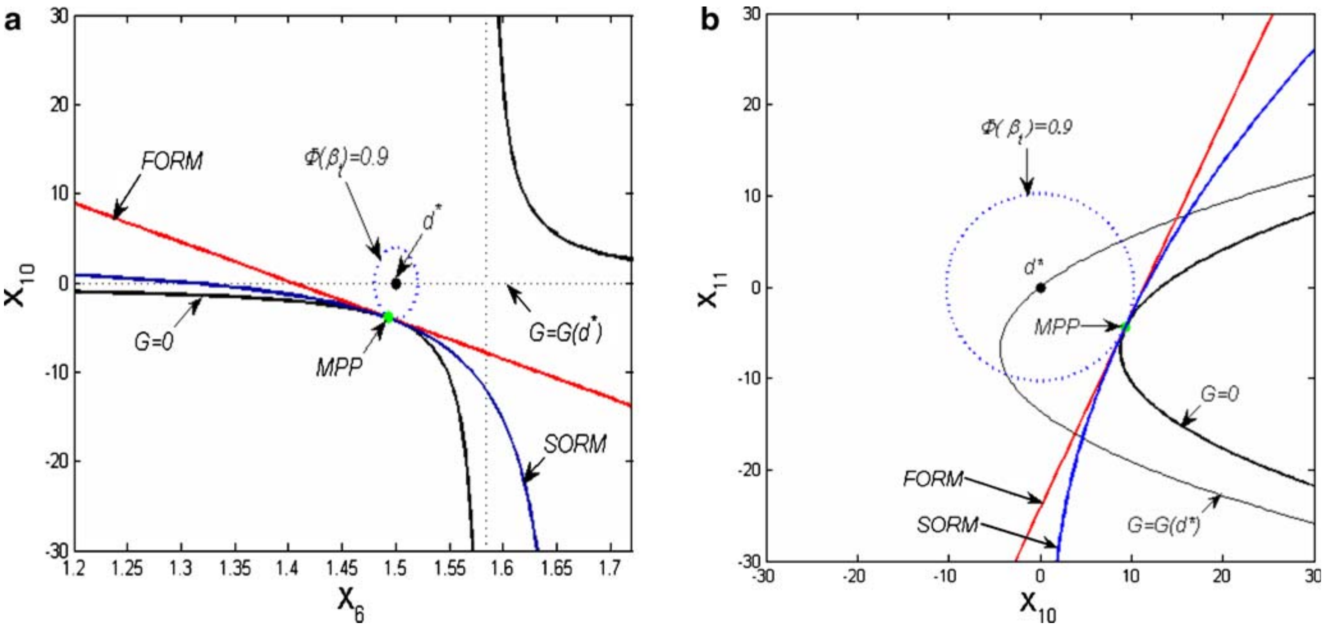


**Fig. 8** **a** FORM and SORM reliability analysis in hyperplane for $G_8$. **b** FORM and SORM reliability analysis in hyperplane for $G_{10}$

Xu 2004a) that converts a multidimensional integration in (28) into multiple one-dimensional integrations. Thus, (28) can be approximated as

$$E[Y^m(\mathbf{X})] \simeq \mathrm{E}\left[\overline{\mathrm{Y}}^{\mathrm{m}}(\mathbf{X})\right] = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \overline{Y}^m \cdot f_{\mathbf{X}}(\mathbf{x}) \cdot d\mathbf{x} \tag{29}$$

where $\overline{Y} = \sum_{j=1}^{n} Y\left(\mu_1, \ldots, \mu_{j-1}, X_j, \mu_{j+1}, \ldots, \mu_n\right) - (n-1) \cdot Y(\mu_1, \ldots, \mu_n)$.

Using a binomial formula, (29) can be evaluated by executing one-dimensional integration recursively. Uncertainty of system responses can be evaluated through multiple one-dimensional numerical integrations. The challenge of the problem still remains how to carry out one-dimensional integration effectively.

To overcome the challenge, the EDR method incorporates three technical components: (1) eigenvector sampling, (2) one-dimensional response approximations for efficient and accurate numerical integration, and (3) a stabilized Pearson system for PDF generation.

**Table 5** The efficiency study for different methods

| Methods | EDR | FORM | SORM | MCS |
|---|---|---|---|---|
| Total number of function evaluation times | 23 | 47 | 47 | 1,000,000 |
| Total number of sensitivity evaluation times | 0 | 47 | 47 | 0 |
| Hessian matrix evaluation times | 0 | 0 | 10 | 0 |

## 4.1 Eigenvector sampling

Accuracy for probability analysis can be increased as the number of integration points becomes larger for the recursive one-dimensional integration. However, the increase of integration points makes simulations prohibitively expensive. To achieve both accuracy and efficiency in probability analysis, one-dimensional response surface will be created using samples along the eigenvectors of a random system. For efficiency, the EDR method employs only either three or five samples along, depending on the nonlinearity of system responses. For $n$ number of random variables, the EDR method demands $2n+1$ or $4n+1$ sample points including the design point.

To obtain the eigenvectors and eigenvalues, an eigenproblem can be formulated as

$$\Sigma X = \lambda X \tag{30}$$

where $\mathbf{X}$ and $\lambda$ are eigenvectors and eigenvalues of the covariance matrix, $\boldsymbol{\Sigma}$. Depending on statistical configura-

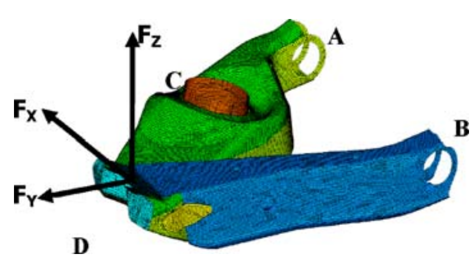


**Fig. 9** Three loading variables (epistemic)

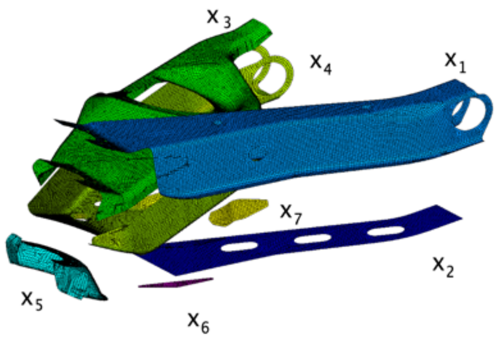


**Fig. 10** Seven thickness variables (aleatory)

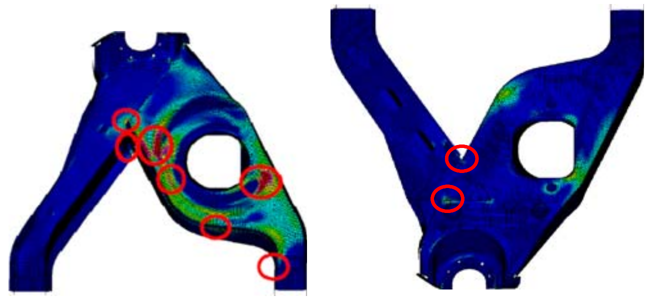

**Fig. 11** Thirty-nine critical constraints of the lower control A-arm model

tion of the system, four different types of problems can be defined: (a) uncorrelated and symmetric, (b) correlated and symmetric, (c) uncorrelated and asymmetric, and (d) correlated and asymmetric. For any circumstance, eigenvector samples will be found at

$$^{1}X_i' = \mu_i - k\sqrt{\lambda_i} \text{ and } ^{2}X_i' = \mu_i + k\sqrt{\lambda_i} \tag{31}$$

where $X_i'$ and $\lambda_i$ are the $i$th eigenvector and eigenvalue and $k$ determines samples along the eigenvectors. The eigenvector samples are used for constructing one-dimensional response approximations using the stepwise moving least squares (SMLS) method in the following section.

## 4.2 SMLS method for numerical integration

The moving least squares (MLS) method (Youn and Choi 2004) is improved by a stepwise selection of basis functions, referred to as the SMLS method. The optimal set of basis terms is adaptively chosen to maximize numerical accuracy by screening the importance of basis terms. This technique is exploited for approximating the integrand in (29). The idea of a stepwise selection of basis functions comes from the stepwise regression method (Myers and Montgomery 1995). The SMLS method allows the increase in the number of numerical integration points without requiring actual system evaluations through simulations or experiments. Thus, a large number of integration points can be used to increase numerical accuracy in assessing statistical moments of the responses while maintaining high efficiency. The EDR method has no restriction to choose numerical integration schemes.

**Table 6** Random properties in lower control A-arm model

| Random variable | Lower bound of mean | Mean | Std. dev. | Dist. type |
|---|---|---|---|---|
| $X_1$ | 0.1 | 0.12 | 0.006 | Normal |
| $X_2$ | 0.1 | 0.12 | 0.006 | Normal |
| $X_3$ | 0.1 | 0.18 | 0.009 | Normal |
| $X_4$ | 0.1 | 0.135 | 0.00675 | Normal |
| $X_5$ | 0.15 | 0.25 | 0.0125 | Normal |
| $X_6$ | 0.1 | 0.18 | 0.009 | Normal |
| $X_7$ | 0.1 | 0.135 | 0.00675 | Normal |

## 4.3 A stabilized Pearson system

The Pearson system (Johnson et al. 1995) can be used to construct the PDF of a random response ($Y$) based on its first four moments (mean, standard deviation, skewness, and kurtosis). The detail expression of the PDF can be achieved by solving the differential equation as

$$\frac{1}{p(Y)}\frac{dp(Y)}{dY} = -\frac{\mathrm{a}+Y}{c_0 + c_1 Y + c_2 Y^2} \tag{32}$$

where $a$, $c_0$, $c_1$, and $c_2$ are four coefficients determined by the first four moments of the random response ($Y$) and expressed as

$$\begin{aligned} c_0 &= (4\beta_2 - 3\beta_1)(10\beta_2 - 12\beta_1 - 18)^{-1}\mu_2 \\ \mathrm{a} = c_1 &= \sqrt{\beta_1}(\beta_2+3)(10\beta_2 - 12\beta_1 - 18)^{-1}\sqrt{\mu_2} \\ c_2 &= (2\beta_2 - 3\beta_1 - 6)(10\beta_2 - 12\beta_1 - 18)^{-1} \end{aligned} \tag{33}$$

where $\beta_1$ is the squares of skewness, $\beta_2$ is the kurtosis, and $\mu_2$ is the variance. The mean value is always treated as zero in the Pearson system, and it can be easily shifted to the true mean value once the differential equation is solved. Basically, the differential equation can be solved based on the different assumptions of the four coefficients $a$, $c_0$, $c_1$,

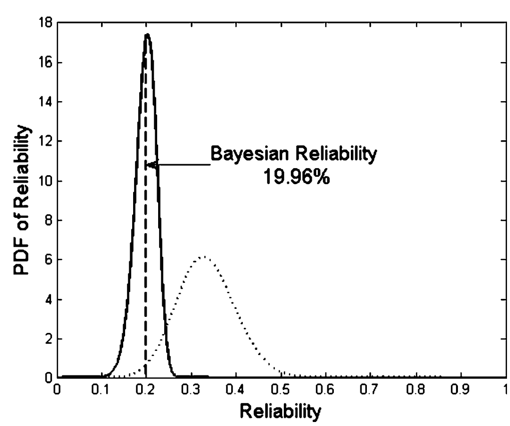


**Fig. 12** Bayesian reliability for $G_1$

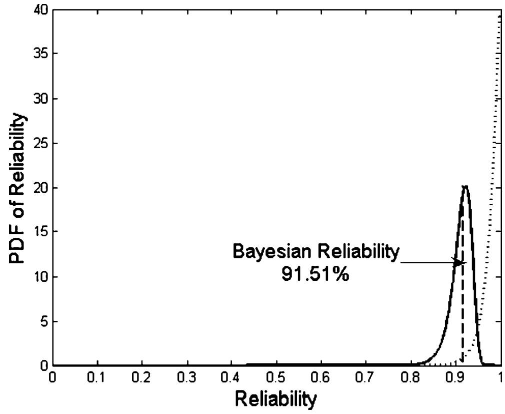


Fig. 13 Bayesian reliability for $G_{24}$

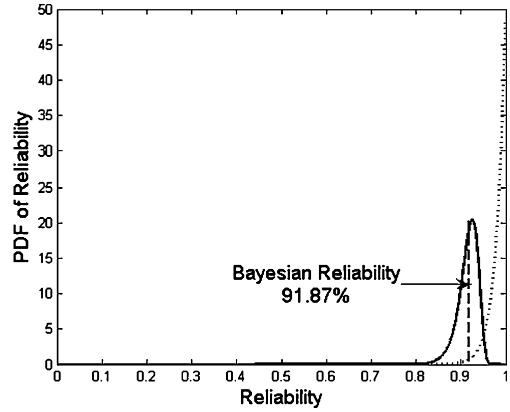


Fig. 15 Bayesian reliability for $G_{38}$

and $c_2$. For example, if $c_1=c_2=0$, this equation can be solved with a normal distribution, and the type 1 in Pearson system corresponds to both roots of $c_0+c_1Y+c_2Y^2$ being real. In the Pearson system, however, a singularity problem is often encountered due to the failure in calculating coefficients of a specific distribution type, which results in numerical instability.

In the EDR method, a stabilized Pearson system is proposed to avoid instability. Two hyper-PDFs (PDF with a perturbed kurtosis value from the original) are generated by fixing the first three statistical moments and incrementally adjusting the original kurtosis by slightly increasing or decreasing the value until two hyper-PDFs are successfully constructed. Then, these two hyper-PDFs are used to approximate the PDF using the original moments. Once the PDF is approximated, the probability of the failure can be evaluated directly. Section 3.4 gives a verification of the EDR method by comparing FORM, SORM, and the EDR method against MCS, for the component reliability analysis of a vehicle side-impact model.

## 4.4 Verification of EDR method

In practice, reliability is the engineering metric to determine how well a product or process is designed. The most common methods for reliability analysis are the first-order reliability method (FORM) and second-order reliability method (SORM), due to their reasonable accuracy and efficiency. This study aims at realizing the feasibility of applying the EDR method towards reliability analysis. Thus, the EDR method, FORM, and SORM will be compared against MCS for an accuracy study through verifying the reliability of vehicle side crash problem.

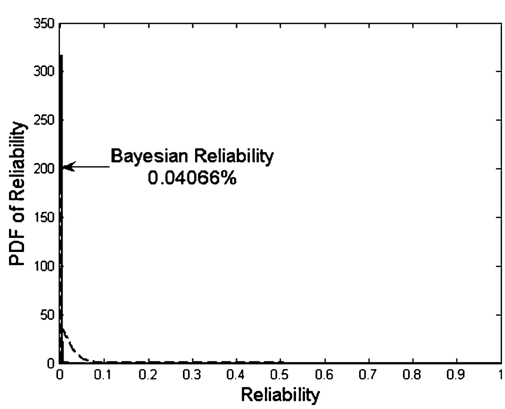


Fig. 14 Bayesian reliability for $G_{35}$

The response surface for the vehicle side-impact model can be briefly summarized with 10 constraints and 11 random variables (Youn et al. 2004). The descriptions of the 11 random variables and the 10 constraints are shown in Tables 2 and 3, respectively. The detailed formulation for this model can be found in Myers and Montomery (1995). In this study, the design used for the reliability verification is obtained from the FORM-based RBDO optimum design with a target component reliability 90%, which is $[d^*]^T=$ [0.5, 1.32739, 0.5, 1.26206, 0.623175, 1.5, 0.5, 0.345, 0.192, 0, 0]. Given this design for the model, component reliabilities will be verified by four different methods: FORM, SORM, EDR method, and Monte Carlo simulation using one million sample points. The comparison results for this study are shown in Table 4.

From Table 4, it is clear that FORM- and SORM-based reliability analysis methods induce substantial error for reliability analysis, whereas the EDR method gives fairly accurate results compared with MCS. Figure 8 shows the first- and second-order approximations in hyperplanes for the failure surfaces of two active constraints, $G_8$ and $G_{10}$, at the given optimum design. From Fig. 8, it is clear that large

Table 7 Verification of optimum designs (MCS with one million samples)

| Method | Optimum points | | Reliability (by MCS) | | |
|---|---|---|---|---|---|
| | $X_1$ | $X_2$ | $G_1$ | $G_2$ | $G_3$ |
| FORM | 3.3786 | 3.1238 | 0.8833 | 0.9170 | 1.0000 |
| EDR | 3.4576 | 3.0898 | 0.9000 | 0.9001 | 1.0000 |

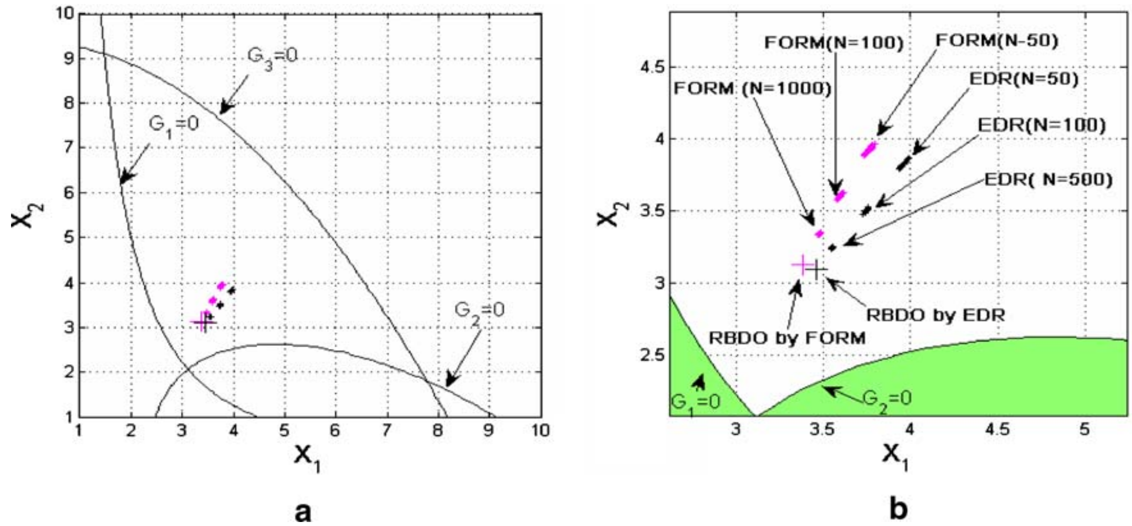


**Fig. 16** Bayesian RBDO by using FORM and EDR with sample size *N*. **a** The optimum designs in the entire design space. **b** The optimum designs (zoomed)

errors may be induced by FORM- and SORM-based reliability analysis methods when the failure surfaces are highly nonlinear. However, these nonlinear behaviors of failure surfaces are accurately captured by the EDR method. Through this study, the feasibility and accuracy using the EDR method for reliability analysis is verified.

It is found in many examples that the EDR method is more accurate than FORM/SORM (Youn et al. 2006b; Youn et al. 2007) because FORM or SORM approximates a failure surface in a linear or quadratic manner at the most probable point (MPP) and then takes an integration over the failure domain, whereas the EDR method precisely approximates the integration directly and the largest error comes from the fourth-order derivative (Rahman and Xu 2004a,b). Moreover, as shown in Table 5, the EDR method turns out to be more efficient than FORM and SORM in this example.

## 5 Bayesian RBDO using EDR method

### 5.1 Bayesian reliability analysis using EDR method

As described in Section 4, the EDR method is more efficient and accurate than FORM and SORM. Because Bayesian RBDO is computationally intensive, it is integrated with the EDR method that evaluates Bayesian reliabilities efficiently and accurately. In this section, Bayesian reliability analysis using the EDR method is performed, considering a lower control arm for the high-mobility, multipurpose, wheeled vehicle (HMMWV).

Vehicle suspension systems experience intense loading conditions throughout their service lives. Control arms act as the backbone of suspension systems, where the majority of the loads are transmitted through. Therefore, it is crucial that control arms be highly reliable while remaining cost-effective. For the purpose of validating the Bayesian RBDO method, a HMMWV lower control arm is presented as a case study. The following example incorporates Bayesian reliability analysis, where a later section shows the use of Bayesian RBDO.

The lower control arm is modeled with plane stress elements using 54,666 nodes, 53,589 elements, and 327,961 degree of freedoms (DOFs), where all welds are modeled using rigid beam elements. For finite element (FE) and design modeling, HyperWorks 7.0 is used. The loading and boundary conditions for this case study are shown in Fig. 9, where loading is applied at the ball joint (point D) in three directions and the boundary conditions are applied at the bushings (points A and B) and the shock-absorber/spring assemble (point C). Due to a lack of data, the loads are considered as epistemic random variables. The design variables for this problem are the thicknesses of the seven major components of the control arm, as shown in Fig. 10. The statistical information of these components, shown in

**Table 8** Efficiency comparison between EDR and FORM

| Methods | EDR | FORM |
|---|---|---|
| Total times of function evaluation | 250 | 1,052 |
| Total times of sensitivity evaluation | 50 | 1,052 |

**Table 9** Assumed random properties for epistemic uncertainties

| Epistemic variable | Distribution |
|---|---|
| $F_x$ | ~*Normal*(1,900, 95) |
| $F_y$ | ~*Normal*(95, 4.75) |
| $F_z$ | ~*Normal*(950, 47.5) |

**Table 10** Random properties in lower control A-arm model

| Random variable | $\mathbf{d}_L$ | $\mu_X=\mathbf{d}$ (mean) | $\mathbf{d}_U$ | Std. dev. | Dist. type |
|---|---|---|---|---|---|
| $X_1$ | 0.1 | 0.120 | 0.5 | 0.00600 | Normal |
| $X_2$ | 0.1 | 0.120 | 0.5 | 0.00600 | Normal |
| $X_3$ | 0.1 | 0.180 | 0.5 | 0.00900 | Normal |
| $X_4$ | 0.1 | 0.135 | 0.5 | 0.00675 | Normal |
| $X_5$ | 0.15 | 0.250 | 0.5 | 0.01250 | Normal |
| $X_6$ | 0.1 | 0.180 | 0.5 | 0.00900 | Normal |
| $X_7$ | 0.1 | 0.135 | 0.5 | 0.00675 | Normal |

Table 6, is well known, and these random parameters are therefore considered as aleatory variables in the Bayesian RBDO.

To determine the hot spots (high-stress concentrations) in the model, which are used to determine the constraints, a worst-case scenario analysis of the control arm is performed. For this worst-case scenario, all the design variables are set at their lower bounds as shown in Table 6, and all the loads are set at their highest values attained from the epistemic data points.

From the worst-case scenario, 39 constraints ($G_1$ to $G_{39}$) are defined on several critical regions using the von Mises stress in Fig. 11. For those constraints, Bayesian reliabilities are defined as

$$R_i^B(\mathbf{X}_a, \mathbf{X}_e; \mathbf{d}) = P_B\left(G_i(\mathbf{X}) = \frac{s_i(\mathbf{X})}{s_U} - 1 \leq 0\right) \tag{34}$$

The PDFs for reliabilities at the critical spots are estimated using Bayesian inference. Four representative PDFs ($G_1$, $G_{24}$, $G_{35}$, and $G_{38}$) are plotted in the dotted curve in Figs. 12, 13, 14, and 15. The extreme distributions (solid curves) of the reliability PDFs are presented in the figures. The median values of the extreme distribution are

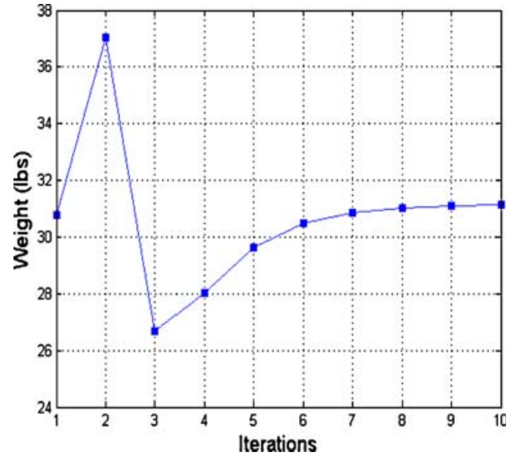

**Fig. 17** Objective function history

then defined as the Bayesian reliabilities for different constraints, which are also plotted in Figs. 12, 13, 14, and 15 as vertical dashed lines. As illustrated in Figs. 11, 12, 13, and 14, $G_1$ and $G_{35}$ (the most critical spots at the current design point) are much less reliable than $G_{24}$ and $G_{38}$. This observation is consistent with a stress contour in Fig. 10 because the stresses in $G_1$ and $G_{35}$ are extremely high. When a target Bayesian reliability is set to 90%, $G_1$ and $G_{35}$ are violated but others are inactive.

### 5.2 Sensitivity analysis for Bayesian reliability

To get the sensitivity of Bayesian reliability with respect to a design variable, a one-to-one mapping between Bayesian reliability and the mean value of the Beta distribution for reliability is used, such as $R_B^i = R_B^i(M_i)$ or $M_i = M_i(R_i^B)$. The transformation between these two variables $R_i^B$ and $M_i$ is shown in (19). Instead of Bayesian reliability, the corresponding mean value of the Beta distribution for reliability can be used for design optimization. Then,

**Table 11** Bayesian RBDO design history

| Iter. | Design | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | $X_1$ | $X_2$ | $X_3$ | $X_4$ | $X_5$ | $X_6$ | $X_7$ | Mass |
| 1 | 0.12 | 0.120 | 0.180 | 0.135 | 0.25 | 0.180 | 0.135 | 30.76 |
| 2 | 0.10 | 0.100 | 0.109 | 0.307 | 0.15 | 0.500 | 0.100 | 37.04 |
| 3 | 0.10 | 0.143 | 0.143 | 0.100 | 0.15 | 0.500 | 0.100 | 26.70 |
| 4 | 0.10 | 0.144 | 0.153 | 0.107 | 0.15 | 0.242 | 0.500 | 28.013 |
| 5 | 0.10 | 0.137 | 0.153 | 0.141 | 0.15 | 0.500 | 0.100 | 29.64 |
| 6 | 0.10 | 0.138 | 0.157 | 0.151 | 0.15 | 0.500 | 0.100 | 30.51 |
| 7 | 0.10 | 0.138 | 0.156 | 0.156 | 0.15 | 0.500 | 0.100 | 30.84 |
| 8 | 0.10 | 0.137 | 0.156 | 0.158 | 0.15 | 0.500 | 0.164 | 31.01 |
| 9 | 0.10 | 0.137 | 0.156 | 0.160 | 0.15 | 0.500 | 0.156 | 31.11 |
| 10 (optimum) | 0.10 | 0.137 | 0.1559 | 0.1598 | 0.15 | 0.500 | 0.177 | 31.13 |

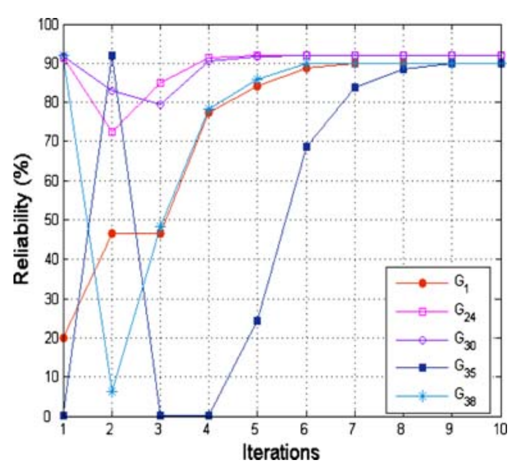

**Fig. 18** Bayesian reliability history

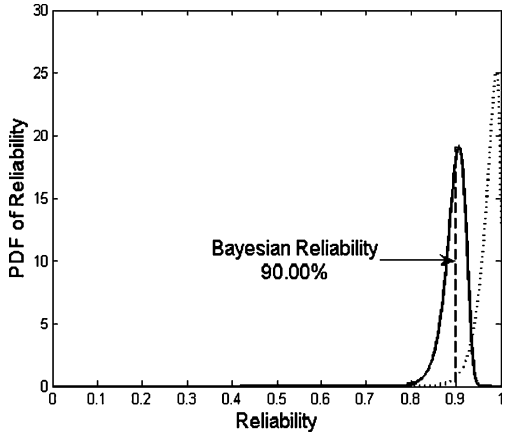


**Fig. 19** Bayesian reliability for $G_1$, $G_{35}$, and $G_{38}$ at the optimum design

Bayesian RBDO can be reformulated with the mean value of the Beta distribution as

$$\begin{array}{ll} \text{minimize} & C(\mathbf{X}_{\mathrm{a}}, \mathbf{X}_e; \mathbf{d}) \\ \text{subject to} & M(R_{B,i}) \geq M_t\left(R^t_{B,i}\right), \quad i = 1, \cdots, np \\ & \mathbf{d}^{\mathrm{L}} \leq \mathbf{d} \leq \mathbf{d}^{\mathrm{U}}, \quad \mathbf{d} \in R^{nd} \text{ and } \mathbf{X}_{\mathrm{a}} \in R^{na}, \mathbf{X}_e \in R^{ne} \end{array} \tag{35}$$

Using (35), sensitivity of the mean value of the Beta distribution for reliability with respect to a design variable must be derived for Bayesian RBDO. Moreover, a target Bayesian reliability, $R^t_B$, must be mapped to the corresponding mean value, $M_t\left(R^t_B\right)$, of the Beta distribution for reliability.

Starting from (22), the sensitivity of the reliability, $R_i$, with respect to the design variable, $d_j$, can be easily computed using the EDR method (Youn et al. 2007; Zhimin et al. 2007).

## 6 Results of BRBDO using EDR method

In this section, two examples are used to demonstrate the proposed methodology of Bayesian RBDO using the EDR method. To make Bayesian RBDO affordable, a distributed computing system is used for this study.

### 6.1 Mathematical example

For Bayesian RBDO, different reliability analysis methods are used to compare numerical accuracy and efficiency: the FORM and the EDR method. Consider the following mathematical problem with three random variables: Two of them are aleatory with $X_i \sim Normal(\mu_i, 0.6)$, $i=1, 2$, and $X_3$ is epistemic with $N$ samples. In this paper, aleatory random variables are considered as design variables, $\mathbf{d}=[d_1, d_2]^T=[\mu_1=\mu(X_1), \mu_2=\mu(X_2)]^T$. Epistemic random variable, $X_3$, is not considered as a design variable because none of its statistical properties are known. Here, the RBDO problem is defined as

$$\begin{array}{ll} \text{Minimize} & d_1 + d_2 \\ \text{Subject to} & P_B(G_i(X) \leq 0) = F^B_{G_i}(0) \geq R^t_{B,i}, \quad i = 1, 2, 3 \\ & 0 \leq d_1 \,\&\, d_2 \leq 10 \end{array} \tag{36}$$

where

$$\begin{array}{l} G_1 = X_1^2 X_2 X_3 / 20 - 1 \\ G_2 = \frac{1}{30}(X_1 + X_2 + X_3 - 6)^2 + \frac{1}{120}(X_1 - X_2 - X_3 - 11)^2 - 1 \\ G_3 = 80 / \left(X_1^2 + 8X_2X_3 + 5\right) - 1 \end{array} \tag{37}$$

In this study, the target reliability is set to $R^t_{B,i} = 90\%$.

Before performing Bayesian RBDO, the EDR method is compared to FORM in terms of numerical accuracy of Frequentist RBDO. The optimum design using the EDR method is quite different from that from RBDO using FORM. These two optimum designs are verified using MCS with one million samples, and the results are summarized in Table 7. It is found that FORM yields an error in reliability estimates due to a linearization of the failure surface at the MPP. Because $G_1$ and $G_2$ at the optimum design are concave and convex, respectively, the reliability for $G_1$ is underestimated, while $G_2$ is overestimated. On the other hand, the optimum design using the EDR method precisely satisfies reliability constraints.

Bayesian RBDO is carried out for different sample sizes with $N$=50, 100, and 500. These samples are randomly generated during the design optimization by assuming $X_3 \sim Normal(1.0, 0.1)$. Different optimum designs will be obtained whenever Bayesian RBDO is performed, even with the same sample size, $N$. This is mainly because an insufficient data size leads to a subjective decision. To understand the subjective decision due to the dearth of data, Bayesian RBDOs for each sample size are performed 20 times using both FORM and the EDR method. Moreover,

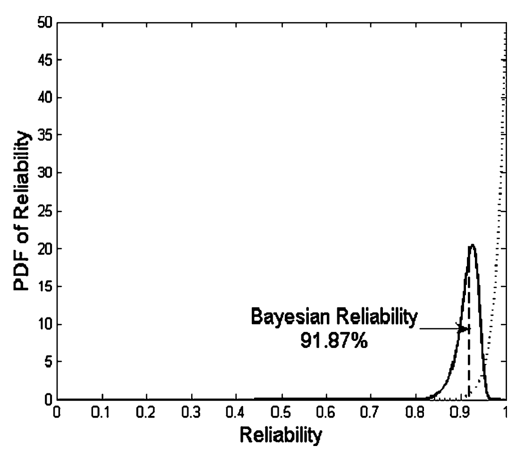


**Fig. 20** Bayesian reliability for $G_{24}$ at the optimum design

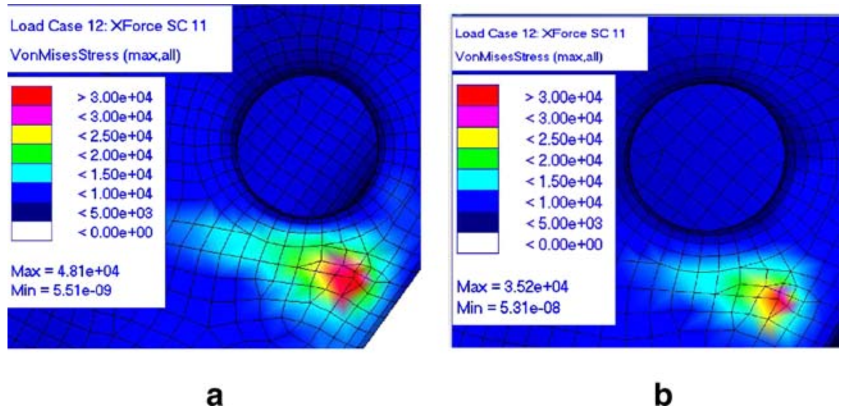


**Fig. 21** Element stress contour for $G_{35}$. **a** At the initial design and **b** at the optimum design

these results are compared to the Frequentist reliability-based optimum design by assuming $X_3$ as aleatory with the statistical properties given above. As expected, both Bayesian RBDO results using FORM and the EDR method asymptotically approach that from the Frequentist results when it increases the sample size of the epistemic variables, as shown in Fig. 15. Bayesian RBDO with the smaller sample size ($N$=50) leads to more subjective decisions. In other words, the optimum designs are more widely scattered. Because of the sufficiency requirement given in Section 3.3, Bayesian RBDO with a smaller sample size yields the optimum designs with greater reliability compared with Frequentist RBDO results. When more than 500 samples are engaged in Bayesian RBDO, it produces the optimum design quite close to that from Frequentist RBDO. The Pareto frontier of the optimum designs can be constructed along the optimum design trajectory as the data size increases, as shown in Fig. 16.

Table 8 shows the total number of function and sensitivity evaluations using FORM and the EDR method in Bayesian RBDO. This example employs 50 data samples for epistemic variables. It is found that the EDR method is much more efficient than FORM. This is because one EDR execution evaluates reliabilities for all constraints, unlike FORM. From this example, it is apparent that the EDR method is much more efficient and accurate than FORM for Bayesian RBDO.

## 6.2 Lower control ARM

The control arm used in Section 4 is used for Bayesian RBDO. In this example, seven thickness design variables are considered as aleatory random variables, whereas three load variables (not design variables) are considered as epistemic random variables. Fifty data sets are employed for the epistemic loads during Bayesian RBDO. These samples are randomly generated using the assumed distributions shown in Table 9. The properties of the design and random variables are shown in Table 10.

With 39 constraints, Bayesian RBDO is formulated as

$$\begin{aligned}&\text{Minimize } \textit{Mass}\\&\text{Subject to } P_B\left(G_i(\mathbf{X}) = \tfrac{s_i(\mathbf{X})}{s_U} - 1 \le 0\right) = F^B_{G_i}(0) \ge \Phi\left(\beta_{t_i}\right),\ i = 1, \cdots, 39\end{aligned} \tag{38}$$

In this study, target reliability is set to $R^t_{B,i} = 90\%$. Ten design iterations reach the Bayesian reliability-based optimum design. The histories of the design parameters, objective function, and the Bayesian reliabilities for significant constraints are shown in Table 11 and Figs. 17

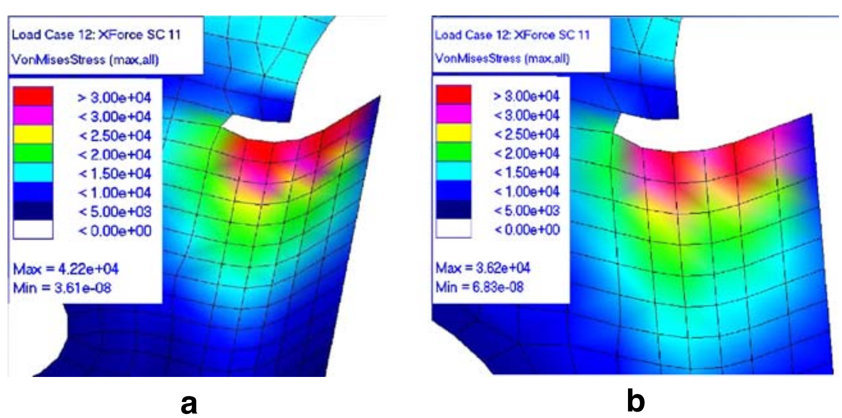


**Fig. 22** Element stress contour for $G_{38}$. **a** At the optimum design and **b** at the initial design

and 18. At the optimum design, three constraints, $G_1$, $G_{35}$, and $G_{38}$, become active and others are feasible. Figures 19 and 20 illustrate the reliability PDFs and Bayesian reliabilities at the optimum design for $G_1$, $G_{24}$, $G_{35}$, and $G_{38}$, of which the PDFs at the initial design are shown in Figs. 12, 13, 14, and 15. The stress contours and the hot spots for the initial design and optimum designs are shown in Figs. 21 and 22.

Finally, the Bayesian reliability-based optimum design is verified by MCS with 10,000 samples. In this verification, three epistemic load variables are assumed to follow the distributions in Table 9. At the optimum design, reliabilities for $G_1$, $G_{35}$, and $G_{38}$ are 98.85, 99.15, and 98.6%. The sufficiency requirement assures higher reliability than the target reliability, 90%.

## 7 Conclusion

Practical engineering analysis and design problems involve both sufficient (aleatory) and insufficient (epistemic) data for their random inputs, such as geometric tolerances, material properties, loads, etc. However, conventional RBDO methods cannot handle the design problems that involve both aleatory and epistemic uncertainties simultaneously. To tackle the design problems engaging both epistemic and aleatory uncertainties, Bayesian RBDO has been proposed. However, Bayesian RBDO becomes extremely expensive when employing either FORM or SORM for reliability prediction. Thus, this paper proposed development of Bayesian RBDO methodology and its integration to a numerical solver, the eigenvector dimension reduction (EDR) method, for Bayesian reliability analysis to improve its efficiency and accuracy. One mathematical example and one engineering design example (vehicle suspension system) were used to demonstrate the feasibility of Bayesian RBDO using EDR method. In Bayesian RBDO using the EDR method, random parameters associated with manufacturing variability are considered as the aleatory random parameters, whereas random parameters associated with the load variability are regarded as the epistemic random parameters. It was found through these two examples that the EDR method enhances numerical efficiency and accuracy for Bayesian RBDO.

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